Proof of Perturbational Duality Between Classical Cavities and Planar Resonators with Magnetic Side-walls

Adham Naji, *Member, IEEE,* and Paul Warr

***Abstract*—In this note a formal proof is given to an observation that the perturbational response of a planar resonator, which can be modelled as a cavity with magnetic side-walls, is the dual of the behaviour of a classical cavity with electric walls. This perturbational duality is observed even though the respective boundary types are not exactly dual. This observation is important as it provides the designer of planar resonators, such as filters and antennas, with a new 'intuition' to the pertubational reaction of such structures, and how it opposes that of classical cavities during perturbation. The formalism of the proof follows clear steps that parallel those found in the classic literature, and the observation is confirmed using numerical methods and measurements.**

***Index Terms*—Perturbation theory, eigenvalue problem, planar resonator, boundary perturbation, microstrip resonator.**

I. Introduction

PERTURBATION calculations are of great importance in the study of classical cavities, as they provide a quick approximate tool for the evaluation of a cavity's resonant frequency shift in response to a relatively small disturbance in its geometric boundaries or its material [1]–[3]. Here we investigate perturbations to the geometric boundaries. In classical cavities, which are made ideally of perfect conductors (electrical-walls), as in Figure 1a, it is known that the frequency shift (relative to the original frequency ω_0) in response to a perturbation is given by

$$\frac{\omega - \omega_0}{\omega_0} \approx \frac{\Delta W_m - \Delta W_e}{W_0} \tag{1}$$

where ΔW_m and ΔW_e are the energies removed by the perturbance from the original resonator volume, and W_0 is the total energy (magnetic and electric) stored in the original resonator before any perturbation.

In planar resonators, which can be modelled as cavities with magnetic side-walls [1]–[3], however, it is observed that the shift direction is the reverse of that dictated by equation (1). In effect, the planar resonator is observed to have a perturbational response that is the dual of that of the classical cavity. In this note, we prove this observation rigorously, and it becomes a new 'intuition' for the designer of planar resonators. Such resonators are extremely common in today's technology, such as microstrip, stripline, and substrate-based filters and patch antennas. This result can also be used in perturbation effect calculations in reconfigurable planar structures, where the insertion of tuning elements constitutes a form of geometric perturbation (e.g. [4]).

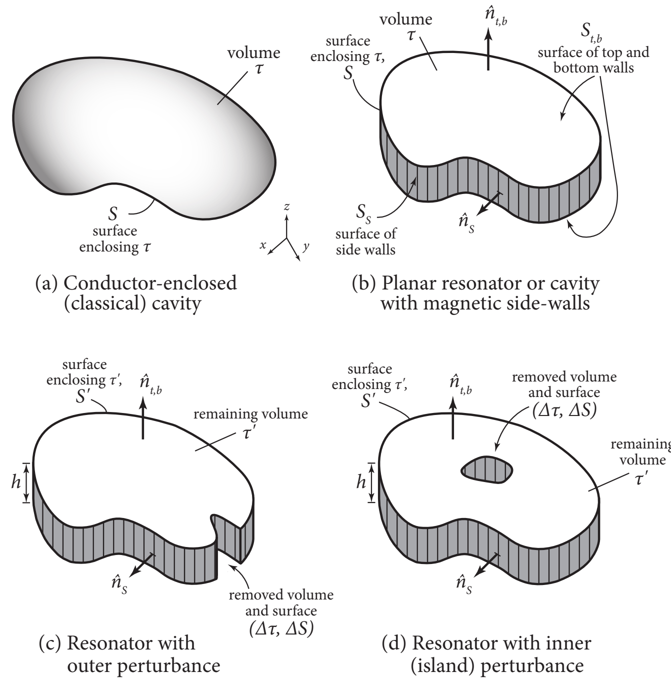

Fig. 1: Illustrations of a classical cavity and a planar resonator, with and without perturbations: (a) the classical conductor-enclosed cavity, (b) the planar resonator or the cavity model with magnetic side-walls and conductive walls (electric-walls) at the top and bottom planes, (c) the planar resonator after an outer perturbation, and (d) the planar resonator after an inner perturbation. The substrate dielectric material is shown as dark-hatched regions, while conductors are shown as white regions. h is the substrate height, τ the original volume, enclosed by surface S, $\Delta\tau$ the perturbation size (removed), ΔS the surface enclosing $\Delta\tau$, and τ' the remaining volume enclosed by the remaining surface S', with $\tau'{=}\tau{-}\Delta\tau$, $S'{=}S{-}\Delta S$, $S{=}S_s{+}S_{t,b}$, $S'{=}S'_s{+}S'_{t,b}$, and $\Delta S{=}\Delta S_s{+}\Delta S_{t,b}$.

It is important to note here that the term *planar* implies no field variations along substrate depth (along z, say), which is much smaller a dimension compared to the resonant wavelength. The latter is in the order of the planar surface dimensions (in the xy plane) of the resonator. Dominant

A. Naji is with the British University in Egypt, Egypt, and also with the Faculty of Engineering, University of Bristol, England. P. Warr is with the Faculty of Engineering, University of Bristol, England; e-mails: adham.naji@ieee.org and paul.a.warr@bristol.ac.uk.

Manuscript received September, 2015.

(lowest-frequency) modes in such structures are assumed to be TM^z_{mn0} type modes. In other words, a planar resonator is effectively two-dimensional (whereas a transmission line resonator, for example, is effectively one-dimensional).

II. Proof Formalism

The problem of finding the resonant (natural) frequency shift in a planar resonator (or a cavity with magnetic side-walls) due to a perturbation in the geometry can follow the same initial steps as those used in perturbation calculations of classic conductor-enclosed cavities (e.g. [1]). However, once the calculation steps reach the stage where the boundary conditions are to be incorporated, the treatment diverges from the classic case, as shown below, due to the mixture of surface boundary conditions. It is shown that the actual perturbational responses (frequency behaviour) of the two cavity types are exactly dual.

With reference to Figure 1, and assuming that the cavity is source-free (i.e. an eigenvalue problem), we can write the field equations before and after perturbation, then attempt to relate them together, to deduce the frequency shift. As in classic literature, we denote quantities before perturbation with a 0-subscript, such as $\bar{E}_0, \bar{H}_0, \omega_0, k_0$, which denote the electric field intensity, magnetic field intensity, angular frequency, and wavenumber in the original resonator. Symbols without such subscript denote the new quantities after perturbation, such as $\bar{E}, \bar{H}, \omega, k$. Barred symbols (e.g. $\bar{A}$) denote vectors, starred symbols (e.g. A^*) denote complex conjugates, $j = \sqrt{-1}$, ϵ is the permittivity, and μ is the permeability. Amplitudes are taken as effective (rms) values, and time dependence as $e^{j\omega t}$.

In the original resonator (Figure 1b), the field equations are derived from Maxwell's equations as

$$-\nabla \times \bar{E}_0 = j\omega_0 \mu \bar{H}_0, \tag{2}$$

$$+\nabla \times \bar{H}_0 = j\omega_0 \epsilon \bar{E}_0. \tag{3}$$

Similarly, after perturbation (Figures 1c,d) the equations are

$$-\nabla \times \bar{E} = j\omega \mu \bar{H}, \tag{4}$$

$$+\nabla \times \bar{H} = j\omega \epsilon \bar{E}. \tag{5}$$

After some inter-manipulation [1], integrating across the perturbed cavity volume (τ') and applying the Divergence Theorem, equations (2)–(5) are combined and reduced to [1]:

$$\begin{aligned} &j(\omega-\omega_0)\iiint_{\tau'}(\epsilon\bar{E}\bar{E}_0^*+\mu\bar{H}\bar{H}_0^*)d\tau \\ &=\iiint_{\tau'}\left[\nabla\cdot(\bar{H}\times\bar{E}_0^*+\nabla\cdot(\bar{H}_0^*\times\bar{E}))\right]d\tau \\ &=\iiint_{\tau'}\nabla\cdot(\bar{H}\times\bar{E}_0^*+\bar{H}_0^*\times\bar{E})d\tau=\oint_{S'}(\bar{H}\times\bar{E}_0^*+\bar{H}_0^*\times\bar{E})d\bar{s}. \end{aligned} \tag{6}$$

Up to this point, the derivation is standard. Now we examine how the magnetic-wall condition on the side walls and electric-wall condition on the top/bottom walls affect the result. It is easier to start by considering the scenario of a disturbance effected in the outer boundary of the cavity, as in Figure 1c. In this scenario, we may decompose the surface integral of (6) to components on the top/bottom surfaces ($S'_{t,b}$) and on the side-walls (S'_s) of the perturbed resonator, giving

$$\begin{aligned} &\oint_{S'}(\bar{H}\times\bar{E}_0^*+\bar{H}_0^*\times\bar{E})d\bar{s}=\oint_{S'}(\bar{H}\times\bar{E}_0^*)\hat{n}ds+\oint_{S'}(\bar{H}_0^*\times\bar{E})\hat{n}ds \\ &=\iint_{\substack{S'_{t,b}\\ \text{(E-wall)}}}(\bar{H}\times\bar{E}_0^*)\hat{n}_{t,b}ds+\iint_{\substack{S'_{s}\\ \text{(M-wall)}}}(\bar{H}\times\bar{E}_0^*)\hat{n}_{s}ds \\ &+\iint_{\substack{S'_{t,b}\\ \text{(E-wall)}}}(\bar{H}_0^*\times\bar{E})\hat{n}_{t,b}ds+\iint_{\substack{S'_{s}\\ \text{(M-wall)}}}(\bar{H}_0^*\times\bar{E})\hat{n}_{s}ds. \end{aligned} \tag{7}$$

But such planar structures will always have the conductors confined to their top or bottom (ground) planes, and the dominant fields will have the electric field direction transverse to them, i.e. TM^z_{mn0}, and $\bar{S}_{t,b} \parallel \bar{S}'_{t,b} \parallel \bar{\Delta} S_{t,b} \parallel \hat{n}_{t,b} \parallel \bar{E} \parallel \bar{E}_0$ (symbol $\parallel$ denotes parallel vectors), we have $\hat{n}_{t,b} \times \bar{E}_0 = 0$ (before perturbation) and $\hat{n}_{t,b} \times \bar{E} = 0$ (after perturbation). On the other hand, on the magnetic-wall of the perturbed resonator, the magnetic field has re-adjusted itself to remain orthogonal to the magnetic-walls on the sides, which will give $\hat{n}_s \times \bar{H} = 0$. Thus, the only term remaining from (7) is the fourth ($\bar{H}_0$ is generally not orthogonal to the S'_s side walls):

$$\oint_{S'}(\bar{H}\times\bar{E}_0^*+\bar{H}_0^*\times\bar{E})d\bar{s}=\iint_{\substack{S'_{s}\\ \text{(M-wall)}}}(\bar{H}_0^*\times\bar{E})\hat{n}_{s}ds. \tag{8}$$

We can re-write this as a closed integral by noting that we can safely take the right hand side as $\oint_{S'}(\bar{H}_0^* \times \bar{E})d\bar{s}$, since $\iint_{S'_{t,b}\,\text{(E-wall)}}(\bar{H}_0^* \times \bar{E})\hat{n}_{t,b}ds = 0$ always.

Since either $\bar{H}_0$ or $\bar{E}$ is orthogonal to the different surfaces that make up S, then $\oint_S(\bar{H}_0^* \times \bar{E})d\bar{s}=0$. We note also that, by keeping the convention of the surface normal vectors $\hat{n}$ pointing outwards, we can reduce the integral domains as follows: $\oint_{S'} \equiv \oint_{S-\Delta S} = 0 - \oint_{\Delta S}$, giving

$$\oint_{S'}(\bar{H}_0^*\times\bar{E})d\bar{s}=-\oint_{\Delta S}(\bar{H}_0^*\times\bar{E})d\bar{s}. \tag{9}$$

We now make the assumption central to perturbation theory, that small disturbances (with smooth and shallow surface distortion) will allow us to consider the $\bar{E}$ and $\bar{H}$ field functions to be unchanged under perturbation, and that the volume integrals over τ' are approximately equal to those over τ [1]–[3]. Poynting's complex power balance gives:

$$\begin{aligned} \oint_{\Delta S}(\bar{E}_0\times\bar{H}_0^*)d\bar{s}&=-j\omega_0\iiint_{\Delta\tau}\left(\mu\left|\bar{H}_0\right|^2-\epsilon\left|\bar{E}_0\right|^2\right)d\tau \\ &=-2j\omega_0(W_m-W_e). \end{aligned} \tag{10}$$

Combining equations (9), (6), and (10) together and solving for $(\omega-\omega_0)/\omega$ gives the final result:

$$\frac{\omega-\omega_0}{\omega_0}\approx\frac{\iiint_{\Delta\tau}\left(\epsilon\left|\bar{E}_0\right|^2-\mu\left|\bar{H}_0\right|^2\right)d\tau}{\iiint_{\tau}\left(\epsilon\left|\bar{E}_0\right|^2+\mu\left|\bar{H}_0\right|^2\right)d\tau}\approx\frac{\Delta W_e-\Delta W_m}{W_0}. \tag{11}$$

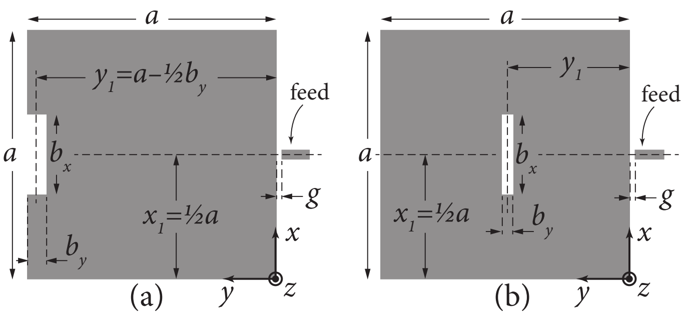

Fig. 2: Two simple perturbation scenarios to validate the theoretical observation, using square stripline planar resonators on $\epsilon_r = 2.17$ and $h = 1.5$ mm substrate (each side) with 17.5 μm copper cladding. One has an E-cut (a) while another has an M-cut (b), with $g = 0.4, a = 20$ and b_x variable in both scenarios, and $b_y = 2$ mm in (a) and 1 mm in (b).

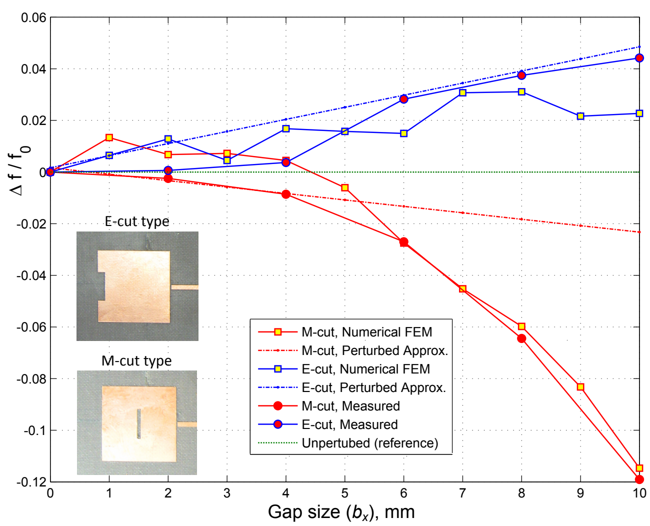

Fig. 3: Results of measurements, FEM, and perturbation approximation, all confirming the predicted behaviour.

Note that the numerator is the inverse of that of the classic conductor-enclosed cavity case expressed in (1), which makes the planar resonator's cavity model the *perturbational dual* of the classic cavity.

The second case of Figure 1d, where an inner perturbance is made (resembling an island), gives the same result. This can be shown by noticing that the first, second, and third terms in the right hand side of equation (7) still vanish because the new magnetic field $\bar{H}$ will position itself to be orthogonal to the gap's inner magnetic-walls as well as the resonator's outer side walls, while the electric field $\bar{E}$ remains transverse to the resonator's plane and is discouraged from occupying the gap (except for a small fringing effect).

Experimental and numerical validation of this observation can be readily carried out by considering a simple structure that is perturbed in two different scenarios, each of which designed in such a way as to cause more disturbance to either the E or the H field energies. To prove the concept it is preferred to use simple structures that have well-defined modes with clear domains for E and H energies and simple perturbation shapes (cuts), which do not cause complex phenomena such as mode-rotation and mode-splitting. We choose the simple square resonator shown in Figure 2. The two types of cuts shown remove mostly E-energy or mostly H-energy without causing mode rotation, since they do not break the shape symmetry with respect to the mode concerned. The dominant mode TM^z_{01} is expressed as $\bar{H}_0 = \hat{x} A \sin ky$ and $\bar{E}_0 = \hat{z} j A \eta \cos ky$, where A is an arbitrary amplitude constant and $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance.

Substituting these equations into equation (11), after integration (variables are defined in Figure 2), gives

$$\frac{\Delta f}{f_0} = b_x \cos 2ky_1 \sin kb_y/(ka^2), \tag{12}$$

which is an approximate model accurate to the first order only (linear with b_x for given b_y, a and y_1), where the perturbation size is small, as expected from a perturbation technique. Measurement results, alongside the predictions given by this approximate perturbation model and by the numerical Finite Element Method (FEM), are shown in Figure 3. We clearly observe how frequency is confirmed to increase with increasing E-cut sizes and decrease with increasing H-cut sizes.

We finally note that a previous general analysis by [5] gave a similar theoretical observation for perturbations that 'pushed-in' outer walls, as in Figures 1c and 2a, whose perturbed surfaces represent simply-connected regions, but did not explicitly include the case of multiply-connected perturbed surfaces (inner cut), such as that in Figures 1d and 2b.

III. Conclusion

In this note it was proved that the geometric perturbational behaviour of a a classic cavity is the dual of that of a planar resonator (or a cavity with magnetic side-walls). This has important implications on approximate (perturbational) calculations of many popular resonating structures and systems of the planar type, such as microstrip, stripline, and substrate-based patches in filters, antennas or reconfigurable structures. More importantly, this simple result provides the designer with an intuition of how the frequency of resonance would shift in a planar resonator, depending on whether the perturbation has largely affected stored electric energy or magnetic energy, and that this shift is opposite to that seen in classic cavities.

Acknowledgment

This work was supported by ITAC project CFP99, Egypt.

References

[1] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*. McGraw-Hill, 1961.

[2] D. M. Pozar, *Microwave Engineering*, 3rd ed. John Wiley & Sons, 2004.

[3] R. E. Collin, *Foundations for Microwave Engineering*, 2nd ed. Wiley Blackwell, 2000.

[4] A. Naji, P. Warr, M. Beach, and K. Morris, "A fundamental limit on the performance of geometrically-tuned planar resonators," *IEEE Transactions on Microwave Theory and Techniques*, vol. 59, no. 6, pp. 1491–1499, Jun. 2011.

[5] P. H. Ladbrooke, " Some Effects of Field Perturbation Upon Cavity-Resonance and Dispersion Measurements on MIC Dielectrics," *IEEE Transactions on Microwave Theory and Techniques*, vol. 25, no. 11, pp. 892–903, Nov. 1977.